Complete Solutions to Nonconvex Fractional Programming Problems

David Yang Gao [1*] and Ning Ruan [1,2†]

David Yang Gao[1] and Ning Ruan[1,2]

1. School of Sciences, Information Technology and Engineering,
University of Ballarat, Ballarat, Vic 3353, Australia.
2. Department of Mathematics and Statistics,
Curtin University, Perth, WA 6845, Australia.

Abstract

This paper presents a canonical dual approach to the problem of minimizing the sum of a quadratic function and the ratio of nonconvex function and quadratic functions, which is a type of non-convex optimization problem subject to an elliptic constraint. We first relax the fractional structure by introducing a family of parametric subproblems. Under certain conditions, we show that the canonical dual of each subproblem becomes a two-dimensional concave maximization problem that exhibits no duality gap. Since the infimum of the optima of the parameterized subproblems leads to a solution to the original problem, we then derive some optimality conditions and existence conditions for finding a global minimizer of the original problem.

Key Words: nonconvex fractional program, sum-of-ratios, global optimization, canonical duality.

*Email: d.gao@ballarat.edu.au
†Email: n.ruan@ballarat.edu.au

1 Introduction

We study in this paper the following nonconvex fractional programming problem:

$$(\mathcal{P}): \qquad \min \left\{ P_0(\mathbf{x}) = f(\mathbf{x}) + \frac{g(\mathbf{x})}{h(\mathbf{x})} \quad : \quad \mathbf{x} \in \mathcal{X} \right\}, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$ and

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x}, \;\; g(\mathbf{x}) = \frac{1}{2}(\frac{1}{2}|B\mathbf{x}|^2 - \lambda)^2, \;\; h(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T H\mathbf{x} - \mathbf{b}^T\mathbf{x},$$

with $B \in \mathbb{R}^{m\times n}$, $Q \in \mathbb{R}^{n\times n}$ being symmetric, $H \in \mathbb{R}^{n\times n}$ negative definite , $\lambda \in \mathbb{R}^+$, and $\mathbf{f}, \mathbf{b} \in \mathbb{R}^n$, where $|v|$ denotes the Euclidean norm of v. Assume that $\mu_0^{-1} = h(H^{-1}\mathbf{b}) > 0$ and $\delta \in (0, \mu_0^{-1}]$, then the feasible domain $\mathcal{X}$ is defined by

$$\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \; h(\mathbf{x}) \geq \delta > 0\},$$

which is a constraint of elliptic type.

Problem $(\mathcal{P})$ belongs to a class of "sum-of-ratios" problems that have been actively studied for several decades. The ratios often stand for efficiency measures representing performance-to-cost, profit-to-revenue or return-to-risk for numerous applications in economics, transportation science, finance, engineering, etc. [1, 6, 10, 16, 18, 19, 23, 25]. Depending on the nature of each application, the functions f, g, h can be affine, convex, concave, or neither. However, even for the simplest case in which f, g, h are all affine functions, problem $(\mathcal{P})$ is still a global optimization problem that may have multiple local optima [5, 22]. In particular, Freund and Jarre [12] showed that the sum-of-ratios problem $(\mathcal{P})$ is NP-complete when f, g are convex and h is concave.

Due to the non-convexity involved in the fractional structure, the ordinary Lagrangean dual only provides a weak duality theorem that may exist a positive duality gap. In this paper, we explore some interesting properties and develop a canonical dual approach based on Gao's work [13] for solving problem $(\mathcal{P})$.

In Section 2, we first parameterize problem $(\mathcal{P})$ into a family of subprograms $\{(\mathcal{P}_\mu)\}$, in which each subproblem is a non-convex quadratic program subject to one quadratic constraint. Then, we show the infimum of the optima of the parameterized subproblems provides a solution to problem $(\mathcal{P})$. Since each subproblem $(\mathcal{P}_\mu)$ is a non-convex problem, a canonical dual problem $(\mathcal{P}_\mu^d)$ is derived. We provide some sufficient conditions to establish both the weak and strong duality theorems (the so called *perfect duality*) for the pair of $(\mathcal{P}_\mu)$ and $(\mathcal{P}_\mu^d)$. In Section 3, we develop some existence conditions under which a global optimizer of the original problem $(\mathcal{P})$ can indeed be identified by solving the corresponding canonical dual problems.

2 Sufficiency for Global Optimality

In order to solve problem $(\mathcal{P})$, we consider the following family of parameterized sub-problem:

$$(\mathcal{P}_\mu): \quad \min \left\{ P_\mu(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x}) \quad : \quad \mathbf{x} \in \mathcal{X}_\mu \right\}, \tag{2}$$

where $\mu \in [\mu_0, \delta^{-1}]$ and

$$\mathcal{X}_\mu = \{\mathbf{x} \in \mathbb{R}^n \mid \; h(\mathbf{x}) \geq \mu^{-1} \geq \delta > 0\}$$

is a convex set. We immediately have the following result:

Lemma 1 *Problem $(\mathcal{P})$ is equivalent to $(\mathcal{P}_\mu)$ in the sense that*

$$\inf_{\mathbf{x}\in\mathcal{X}} P_0(\mathbf{x}) = \inf_{\mu\in[\mu_0,\delta^{-1}]} \inf_{\mathbf{x}\in\mathcal{X}_\mu} P_\mu(\mathbf{x}). \tag{3}$$

Proof. It is easy to see that

$$\begin{aligned}
& \inf_{\mathbf{x}\in\mathcal{X}} P_0(\mathbf{x}) \\
= \; & \inf_{\mathbf{x}\in\mathcal{X}} \left\{ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \frac{g(\mathbf{x})}{h(\mathbf{x})} \right\} \\
= \; & \inf_{\mu\in[\mu_0,\delta^{-1}]} \; \inf_{h(\mathbf{x})=\mu^{-1}} \left\{ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \frac{g(\mathbf{x})}{h(\mathbf{x})} \right\} \\
= \; & \inf_{\mu\in[\mu_0,\delta^{-1}]} \; \inf_{h(\mathbf{x})=\mu^{-1}} \left\{ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x}) \right\} \\
\geq \; & \inf_{\mu\in[\mu_0,\delta^{-1}]} \; \inf_{\mathbf{x}\in\mathcal{X}_\mu} \left\{ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x}) \right\} \\
= \; & \inf_{\mu\in[\mu_0,\delta^{-1}]} \; \inf_{\mathbf{x}\in\mathcal{X}_\mu} P_\mu(\mathbf{x}).
\end{aligned}$$

Conversely,

$$\begin{aligned}
& \inf_{\mu\in[\mu_0,\delta^{-1}]} \; \inf_{\mathbf{x}\in\mathcal{X}_\mu} \left\{ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x}) \right\} \\
= \; & \inf_{\mu\in[\mu_0,\delta^{-1}]} \; \inf_{h(\mathbf{x})\geq\mu^{-1}} \left\{ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu g(\mathbf{x}) \right\} \\
\geq \; & \inf_{\mu\in[\mu_0,\delta^{-1}]} \; \inf_{h(\mathbf{x})\geq\mu^{-1}} \left\{ \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \frac{g(\mathbf{x})}{h(\mathbf{x})} \right\} \text{ (since } g(\mathbf{x}) > 0) \\
= \; & \inf_{\mathbf{x}\in\mathcal{X}} P_0(\mathbf{x}).
\end{aligned}$$

This completes the proof of the lemma. $\square$

Now, for any $\mu \in [\mu_0, \delta^{-1}]$, we define

$$G_\mu(\varsigma, \sigma) = Q + \mu B^T B \varsigma - \sigma H, for\ \varsigma \geq -\lambda,\ \sigma \geq 0, \tag{4}$$
$$\mathcal{S}_\mu^+ = \{\varsigma \geq -\lambda,\ \sigma \geq 0 |\ G_\mu(\varsigma, \sigma) \succ 0\}, \tag{5}$$

where '$\succ$' means positive definiteness of a matrix. Then, the parametrical canonical dual problem can be proposed as the following:

$$P_\mu^d(\varsigma, \sigma) = -\frac{1}{2}(\mathbf{f} - \sigma \mathbf{b})^T G_\mu^{-1}(\varsigma, \sigma)(\mathbf{f} - \sigma \mathbf{b}) - \mu \lambda \varsigma - \frac{\mu}{2}\varsigma^2 + \frac{\sigma}{\mu}. \tag{6}$$

Given any $\mu \in [\mu_0, \delta^{-1}]$, consider the following canonical dual problem $(\mathcal{P}_\mu^d)$:

$$(\mathcal{P}_\mu^d) \qquad \sup \quad P_\mu^d(\varsigma, \sigma)$$
$$\text{s.t.} \quad (\varsigma, \sigma) \in \mathcal{S}_\mu^+.$$

Theorem 1 *(Weak Duality) If there exists a global maximizer* $(\varsigma_\mu, \sigma_\mu)$ *of* $P_\mu^d(\varsigma, \sigma)$ *over* $\mathcal{S}_\mu^+$*, then the vector*

$$\mathbf{x}_\mu = G_\mu^{-1}(\varsigma_{\mu,}, \sigma_\mu)(\mathbf{f} - \sigma_\mu \mathbf{b}) \tag{7}$$

is a global minimizer of $(\mathcal{P}_\mu)$ *over* $\mathcal{X}_\mu$ *and*

$$P_\mu^d(\varsigma, \sigma) \leq P_\mu(\mathbf{x}),\ \forall (\mathbf{x}, \varsigma, \sigma) \in \mathcal{X}_\mu \times \mathcal{S}_\mu^+. \tag{8}$$

Proof. Let $\Lambda(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be the *geometrical transformation*[13, 14, 15] defined by

$$\xi = \Lambda(\mathbf{x}) = \frac{1}{2}|B\mathbf{x}|^2 - \lambda \tag{9}$$

and let

$$U(\xi) = \frac{1}{2}\xi^2 \tag{10}$$

Then, Problem $(\mathcal{P}_\mu)$ in (2) can be written as the following unconstrained optimization problem

$$\min \left\{ P(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q \mathbf{x} - \mathbf{f}^T \mathbf{x} + \mu(U(\Lambda(\mathbf{x})) - \sigma(h(\mathbf{x}) - \mu^{-1}) | \mathbf{x} \in \mathbb{R}^n. \right\}. \tag{11}$$

Let ς be the dual variable of ξ, i.e., $\varsigma = \nabla U(\xi) = \xi$, the Legendre conjugate $U^*(\varsigma)$ can be uniquely defined by

$$U^\sharp(\varsigma) = \text{sta}_{\xi \geq \lambda}\{\xi \varsigma - U(\xi)\} = \frac{1}{2}\varsigma^2 \tag{12}$$

where $\varsigma \in \mathcal{V}_a^* = \{\varsigma \in \mathbb{R} | \varsigma \geq -\lambda\}$.

By replacing $U(\Lambda(\mathbf{x}))$ with $\Lambda(\mathbf{x})^T\varsigma - U^\sharp(\varsigma)$ in (11), we define the *total complementary function* as

$$\begin{aligned}
\Xi(\mathbf{x},\varsigma,\sigma) &= \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu(U(\Lambda(\mathbf{x})) - \sigma(\frac{1}{2}\mathbf{x}^T H\mathbf{x} - \mathbf{b}^T\mathbf{x} - \mu^{-1}) \\
&= \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu(\Lambda(\mathbf{x})^T\varsigma - U^\sharp(\varsigma)) - \sigma(\frac{1}{2}\mathbf{x}^T H\mathbf{x} - \mathbf{b}^T\mathbf{x} - \mu^{-1}) \\
&= \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \mathbf{f}^T\mathbf{x} + \mu[(\frac{1}{2}(|B\mathbf{x}|^2 - \lambda)\varsigma - \frac{1}{2}\varsigma^2] - \sigma(\frac{1}{2}\mathbf{x}^T H\mathbf{x} - \mathbf{b}^T\mathbf{x} - \mu^{-1}) \\
&= \frac{1}{2}\mathbf{x}^T G_\mu(\varsigma,\sigma)\mathbf{x} - (\mathbf{f} - \sigma\mathbf{b})^T\mathbf{x} - \mu\lambda\varsigma - \frac{\mu}{2}\varsigma^2 + \frac{\sigma}{\mu},
\end{aligned}$$

where $G_\mu(\varsigma,\sigma)$ is defined in (4). Note that $\Xi(\mathbf{x},\varsigma,\sigma)$ is convex in $\mathbf{x} \in \mathbb{R}^n$ for any given $(\varsigma,\sigma) \in \mathcal{S}_\mu^+$ and affine (hence concave) in (ς,σ) for any given $\mathbf{x} \in \mathbb{R}^n$. By the criticality condition

$$\frac{\partial \Xi}{\partial \mathbf{x}} = G_\mu(\varsigma,\sigma)\mathbf{x} - (\mathbf{f} - \sigma\mathbf{b}) = 0. \tag{13}$$

we have $\mathbf{x}(\sigma) = G_\mu^{-1}(\varsigma,\sigma)(\mathbf{f} - \sigma\mathbf{b})$, which is the global minimizer of $\Xi(\mathbf{x},\sigma)$. Moreover,

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^n} \Xi(\mathbf{x},\varsigma,\sigma) &= \Xi(\mathbf{x}(\varsigma,\sigma),\varsigma,\sigma) \\
&= \frac{1}{2}\mathbf{x}(\varsigma,\sigma)^T(G_\mu(\varsigma,\sigma))\mathbf{x}(\varsigma,\sigma) - (\mathbf{f} - \sigma\mathbf{b})^T\mathbf{x}(\varsigma,\sigma) - \mu\lambda\varsigma - \frac{\mu}{2}\varsigma^2 + \frac{\sigma}{\mu} \\
&= \frac{1}{2}\mathbf{x}(\varsigma,\sigma)^T(\mathbf{f} - \sigma\mathbf{b}) - (\mathbf{f} - \sigma\mathbf{b})^T\mathbf{x}(\varsigma,\sigma) - \mu\lambda\varsigma - \frac{\mu}{2}\varsigma^2 + \frac{\sigma}{\mu} \\
&= -\frac{1}{2}(\mathbf{f} - \sigma\mathbf{b})^T\mathbf{x}(\varsigma,\sigma) - \mu\lambda\varsigma - \frac{\mu}{2}\varsigma^2 + \frac{\sigma}{\mu} \\
&= -\frac{1}{2}(\mathbf{f} - \sigma\mathbf{b})^T G_\mu^{-1}(\varsigma,\sigma)(\mathbf{f} - \sigma\mathbf{b}) - \mu\lambda\varsigma - \frac{\mu}{2}\varsigma^2 + \frac{\sigma}{\mu} \\
&= P_\mu^d(\varsigma,\sigma).
\end{aligned}$$

By the assumption, $(\varsigma_\mu,\sigma_{mu})$ is a global maximizer of $P_\mu^d(\varsigma,\sigma)$. If $(\varsigma_\mu,\sigma_\mu)$ is an interior of $\mathcal{S}_\mu^+$, then $\frac{\partial}{\partial\varsigma}P_\mu^d(\varsigma_\mu,\sigma_\mu) = 0$, and $\frac{\partial}{\partial\sigma}P_\mu^d(\varsigma_\mu,\sigma_\mu) = 0$. Otherwise, we have $\frac{\partial}{\partial\varsigma}P_\mu^d(\varsigma_\mu,\sigma_\mu) = 0$, $\sigma_\mu = 0$, $\frac{\partial}{\partial\sigma}P_\mu^d(\varsigma_\mu,\sigma_\mu) \leq 0$. In either case, If we denote $\mathbf{x}_\mu = \mathbf{x}(\varsigma_\mu,\sigma_\mu) = G_\mu^{-1}(\varsigma_\mu,\sigma_\mu)(\mathbf{f} - \sigma_\mu\mathbf{b})$, we have

$$\begin{aligned}
\frac{\partial}{\partial\varsigma}P_\mu^d(\varsigma_\mu,\sigma_\mu) &= \mu(\frac{1}{2}\mathbf{x}(\varsigma_\mu,\sigma_\mu)^T B^T B\mathbf{x}(\varsigma,\sigma) - \lambda - \varsigma) = 0, \\
\frac{\partial}{\partial\sigma}P_\mu^d(\varsigma_\mu,\sigma_\mu) &= \frac{1}{\mu} - \mathbf{x}_\mu^T(\frac{1}{2}H\mathbf{x}_\mu - \mathbf{b}) \leq 0.
\end{aligned}$$

That is,

$$\begin{aligned}
&\varsigma = \frac{1}{2}|B\mathbf{x}|^2 - \lambda, \\
&\frac{1}{2}\mathbf{x}^T H\mathbf{x} - \mathbf{b}^T\mathbf{x} - \mu^{-1} \geq 0.
\end{aligned}$$

Therefore, $\mathbf{x}_\mu \in \mathcal{X}_\mu$, and for any $(\varsigma, \sigma) \in \mathcal{S}_\mu^+$, we have

$$\begin{aligned}
P_\mu^d(\varsigma, \sigma) &\leq P_\mu^d(\varsigma_\mu, \sigma_\mu) \\
&= \min_{\mathbf{x} \in \mathbb{R}^n} \Xi(\mathbf{x}, \varsigma_\mu, \sigma_\mu) \\
&= \Xi(\mathbf{x}_\mu, \varsigma_\mu, \sigma_\mu) \\
&= \min_{\mathbf{x} \in \mathcal{X}_\mu} \Xi(\mathbf{x}, \varsigma_\mu, \sigma_\mu) \\
&= \frac{1}{2}\mathbf{x}^T Q \mathbf{x} - \mathbf{f}^T \mathbf{x} + \mu(\Lambda(\mathbf{x})^T \varsigma - U^\sharp(\varsigma)) - \sigma(\frac{1}{2}\mathbf{x}^T H \mathbf{x} - \mathbf{b}^T \mathbf{x} - \mu^{-1}) \\
&\leq \frac{1}{2}\mathbf{x}^T Q \mathbf{x} - \mathbf{f}^T \mathbf{x} + \mu(\Lambda(\mathbf{x})^T \varsigma - U^\sharp(\varsigma)) \\
&= \frac{1}{2}\mathbf{x}^T Q \mathbf{x} - \mathbf{f}^T \mathbf{x} + \mu(\frac{1}{2}(\frac{1}{2}|Bx|^2 - \lambda)^2 = P_\mu(\mathbf{x}).
\end{aligned}$$

This completes the proof. □

Theorem 2 *(Strong Duality) If $(\varsigma_\mu, \sigma_\mu)$ is a critical point of $P_\mu^d(\varsigma, \sigma)$ over $\mathcal{S}_\mu^+$, then $(\mathcal{P}_\mu^d)$ is perfectly dual to $(\mathcal{P}_\mu)$ in the sense that the vector*

$$\mathbf{x}_\mu = G_\mu^{-1}(\varsigma_\mu, \sigma_\mu)(\mathbf{f} - \sigma_\mu \mathbf{b}) \tag{14}$$

is a global minimizer of $(\mathcal{P}_\mu)$ and $(\varsigma_\mu, \sigma_\mu)$ is a global maximizer of $(\mathcal{P}_\mu^d)$, and

$$\min_{\mathbf{x} \in \mathcal{X}_\mu} P_\mu(\mathbf{x}) = P_\mu(\mathbf{x}_\mu) = P_\mu^d(\varsigma_\mu, \sigma_\mu) = \max_{(\varsigma, \sigma) \in \mathcal{S}_\mu^+} P_\mu^d(\varsigma, \sigma). \tag{15}$$

Proof. The proof basically follows that of the former weak duality Theorem, The only difference lies in the assumption that $(\varsigma_\mu, \sigma_\mu)$ is a critical point of $P_\mu^d(\varsigma, \sigma)$ over $\mathcal{S}_\mu^+$. In this case, $\frac{\partial}{\partial \varsigma} P_\mu^d(\varsigma_\mu, \sigma_\mu) = 0$, and $\frac{\partial}{\partial \sigma} P_\mu^d(\varsigma_\mu, \sigma_\mu) = 0$. So $\mathbf{x}_\mu = \mathbf{x}(\varsigma_\mu, \sigma_\mu) = G_\mu^{-1}(\varsigma_\mu, \sigma_\mu)(\mathbf{f} - \sigma_\mu \mathbf{b})$ is on the boundary of $\mathcal{X}_\mu$. That is, $\frac{1}{2}\mathbf{x}^T H \mathbf{x} - \mathbf{b}^T \mathbf{x} - \mu_{-1} = 0$. this further implies that

$$P_\mu^d(\varsigma_\mu, \sigma_\mu) = \Xi(\mathbf{x}_\mu, \varsigma_\mu, \sigma_\mu) = P_\mu(\mathbf{x}_\mu) \tag{16}$$

and the equation (15) follows naturally. □

The above results immediately lead to the following sufficient condition for finding the global optimizer of problem $(\mathcal{P})$:

Corollary 1 *If $(\varsigma_\mu, \sigma_\mu) \in \mathcal{S}_\mu^+$ holds for all $\mu \in [\mu_0, \delta^{-1}]$, then*

$$\min_{\mathbf{x} \in \mathcal{X}} P_0(\mathbf{x}) = \min_{\mu \in [\mu_0, \delta^{-1}]} P_\mu^d(\varsigma_\mu, \sigma_\mu). \tag{17}$$

3 Existence of Global Optimality

Before we provide the condition for the existence of a global optimal solution $(\varsigma_\mu, \sigma_\mu)$ to problem (P_μ^d) over $\mathcal{S}_\mu^+$ with any given $\mu \in [\mu_0, \delta^{-1}]$, we need the following property.

Lemma 2 *For any $\mu \in [\mu_0, \delta^{-1}]$, $P_\mu^d(\sigma, \varsigma)$ is a two-dimensional concave function over $\mathcal{S}_\mu^+$.*

Proof. Notice that the Hessian Matrix of the dual objective function is

$$\nabla^2 P^d(\sigma, \varsigma) = S = \begin{pmatrix} H_{\sigma^2} & H_{\sigma\varsigma} \\ H_{\varsigma\sigma} & H_{\varsigma^2} \end{pmatrix},$$

where

$$\begin{aligned} H_{\sigma^2} &= -(Hx(\varsigma, \sigma) - b)G^{-1}(\boldsymbol{\sigma}, \sigma)(H\mathbf{x}(\boldsymbol{\sigma}, \sigma) - b), \\ H_{\varsigma^2} &= -(\mu B^T B)^2 G^{-1}(\varsigma, \sigma)\mathbf{x}(\varsigma, \sigma) - \mu I, \\ H_{\varsigma\sigma} &= \mu B^T B\mathbf{x}(\varsigma, \sigma)G^{-1}(\varsigma, \sigma)(H\mathbf{x}(\varsigma, \sigma) - b). \end{aligned}$$

In order to show the dual function is a concave function, it is equivalent to show that

$$S_0 = \begin{pmatrix} H_{\sigma^2} & H_{\sigma\varsigma} \\ H_{\varsigma\sigma} & H_{\varsigma^2} + \mu \end{pmatrix},$$

is semi-negative definite. By Sylvester's Criterion, it suffices to show that all the leading principal minors have a non-positive determinant. Obviously, the first $n-1$ leading principal minors have non-positive determinants, since

$$-(H\mathbf{x}(\varsigma, \sigma) - b)G^{-1}(\varsigma, \sigma)(H\mathbf{x}(\varsigma, \sigma) - b) \tag{18}$$

is semi-negative definite. It is left to show $det(S_0) \le 0$. Note that

$$\begin{aligned} S_0 &= C \cdot D \\ &= \begin{pmatrix} -(H\mathbf{x}(\varsigma, \sigma) - b)G^{-1}(\varsigma, \sigma) & 0 \\ 0 & UB^T BG^{-1}(\varsigma, \sigma)\mathbf{x}(\varsigma, \sigma) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} H\mathbf{x}(\varsigma, \sigma) - b & UB^T B \\ H\mathbf{x}(\varsigma, \sigma) - b & UB^T B \end{pmatrix} \end{aligned}$$

Apparently, $Rank(C \cdot D) \le Rank(D) \le n$. We can make a conclusion that $det S_0 = 0$. Thus, S is semi-negative definite, which implies that dual function is concave function.

$\square$

Let $\partial\mathcal{S}_\mu^+$ denotes a singular hyper-surface defined by

$$\partial\mathcal{S}_\mu^+ = \{\boldsymbol{\sigma} \geq -\lambda, \sigma \geq 0 | \; G_\mu(\varsigma,\sigma) \succeq 0, \; \det G_\mu(\varsigma,\sigma) = 0\}. \tag{19}$$

Theorem 3 *(Existence) Given any $\mu \in [\mu_0, \delta^{-1}]$, if*

$$\lim_{(\varsigma,\,\sigma)\to\partial\mathcal{S}_a^+} P_\mu^d(\varsigma,\sigma) = -\infty, \;\; \forall(\varsigma,\sigma) \in \mathcal{S}_\mu^+, \tag{20}$$

and

$$\lim_{\varsigma\to\infty,\,\sigma\to-\infty} P_\mu^d(\varsigma,\sigma) = -\infty, \;\; \forall(\varsigma,\sigma) \in \mathcal{S}_\mu^+, \tag{21}$$

then the canonical dual problem $(\mathcal{P}_\mu^d)$ has at least one global optimal solution $(\varsigma_\mu, \sigma_\mu) \in \mathcal{S}_\mu^+$.

Proof. It follows from (20) and (21) that there exists one $(\varsigma_\mu, \sigma_\mu) \in \mathcal{S}_\mu^+$ to be a critical point of $P_\mu^d(\varsigma,\sigma)$. By Theorem 2, we know $(\varsigma_\mu, \sigma_\mu)$ is a global maximizer of $(\mathcal{P}_\mu^d)$. $\square$

4 Conclusions

In this paper, we study a kind of problems with sum of a quadratic function and the ratio of nonconvex function and quadratic function as its objective function. We first parameterize such a problem into a family of subproblems. Then we develop a corresponding canonical duality theory, both in weak and strong duality form, to handle each subproblem. Based on the properties of the subproblems, we provide non only the extremality conditions for global optimality of the original problem, but also existence conditions to assure that the global optimal solutions of the primal problems can indeed be found by solving a sequence of concave maximization problems.

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